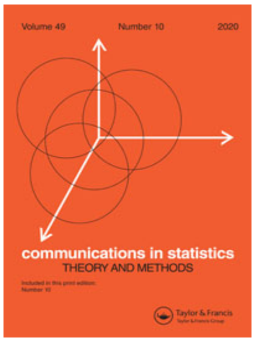

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On estimating the reliability in a multicomponent stress-strength model based on Chen distribution

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ABSTRACT
In this article, we obtain point and interval estimates of multicomponent stress-strength reliability model of an *s*-out-of-*j* system using classical and Bayesian approaches by assuming both stress and strength variables follow a Chen distribution with a common shape parameter which may be known or unknown. The uniformly minimum variance unbiased estimator of reliability is obtained analytically when the common parameter is known. The behavior of proposed reliability estimates is studied using the estimated risks through Monte Carlo simulations and comments are obtained. Finally, a data set is analyzed for illustrative purposes.

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1. Introduction

In the literature, the term stress-strength in context of reliability was initially introduced by Church and Harris (1970). Since then several authors have discussed single-component stress-strength models by taking into consideration of various lifetime distributions. The stress-strength model defines the reliability R of a component as the probability that strength of a unit (X) is greater than the stress (Z) imposed on it, that is, $R = P(X>Z)$. Estimation of stress-strength reliability has attracted some attention among researchers. We refer to Kotz, Lumelskii, and Pensky (2003) for different applications of such estimation problems. The stress-strength plays an important role in reliability analysis, for example, if X denotes the strength of a system which is subjected to a stress Z, then the reliability R measures the system performance and it is very common in the context of mechanical reliability of a system. Note that a system fails whenever the applied stress becomes greater than its strength.

A system having more than one component is called a multicomponent system. A multicomponent system may be a series system, a parallel system, or some combinations of these two systems. A bridge structure is an example of a multicomponent system. The series and parallel systems are special cases of a more general class of systems referred to as s-out-of-j system. A system belonging to this class can be one of following two types: (i) A system that fails with the failure of the s-th component, denoted by s-out-of-j:F

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system; (*ii*) A system that functions as long as at least s components work and is denoted by s-out-of-j:G system, where $1 \leq s \leq j$. Multicomponent systems have been found wide applications in both industrial and military operations. For example, in a communications system with three transmitters, the average message load may be such that at least two transmitters must be operational at all times otherwise critical messages may be lost. Thus, the transmission subsystem functions as a 2-out-of-3:G system.

Suppose a multicomponent system which has j independent and identically distributed strength components and each component is exposed to a common random stress is considered. The system works only when at least s out of j $(1 \leq s \leq j)$ strengths exceed the stress. This corresponds to an s-out-of-j:G system. Let $X_1, ..., X_j$ be the strength variables from the cumulative distribution function (CDF) $F(\cdot)$ and Z be the common stress variable from the CDF $G(\cdot)$. Then, the reliability of a multicomponent system is given by

$$\begin{aligned} R_{s,j} &= P\big(\text{at least } s \text{ of the } (X_1, X_2, ..., X_j) \text{ exceed } Z\big) \\ &= \sum_{i=s}^{j} \binom{j}{i} \int_{-\infty}^{\infty} [1 - F(z)]^i [F(z)]^{j-i} dG(z) \end{aligned}$$

The reliability in a multicomponent stress-strength model was initially studied by Bhattacharyya and Johnson (1974). Since then, several authors used different classical procedures to estimate the reliability in multicomponent stress-strength models (see, for instance, Rao and Kantam (2010), Rao (2012a, 2012b, 2012c), and Rao et al. (2013, 2015) when the underlying distributions follow log-logistic, generalized exponential, generalized inverted exponential, Rayleigh, Burr Type XII, and exponentiated Weibull distributions. At recent past, Kizilaslan and Nadar (2015, 2018), Kizilaslan (2017, 2018), and Dey, Mazucheli, and Anis (2017) considered estimation of reliability in multicomponent stress-strength models under classical and Bayesian framework for Weibull, bivariate Kumaraswamy, general class of inverse exponentiated distributions, and Kumaraswamy distributions.

The main attempt of this article is to obtain estimates of $R_{s,j}$ using classical and Bayesian approaches when strength and stress variable are independent and follow Chen distributions. We organize the rest of this article as follows. In Section 2, the system model and associated reliability are discussed. In Section 3, we obtain the maximum likelihood estimates (MLEs) and also compute the Fisher information matrix. Estimation of $R_{s,j}$ is considered in Section 4 under the assumption that the common shape parameter may be known or unknown. For the point estimation, we obtain the MLE, uniformly minimum variance unbiased estimator (UMVUE), and Bayes estimator. For the interval estimation, we construct the asymptotic confidence interval and the highest posterior density (HPD) credible interval. We conduct a Monte Carlo simulation study in Section 5 to compare the proposed estimates of $R_{s,j}$. A real data set analysis is presented in Section 6. Finally, we conclude the article in Section 7.

2. Model description and reliability of the system

Chen (2000) proposed a new two parameter lifetime distribution with bathtub shaped or increasing failure rate function. Its probability density function (PDF) is given by

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}\exp\left(\alpha\left(1-e^{x^{\beta}}\right)+x^{\beta}\right),\quad x>0$$

and the corresponding CDF is

$$F(x;\alpha,\beta)=1-\exp\left(\alpha\left(1-e^{x^{\beta}}\right)\right),\quad x>0 \tag{1}$$

where $\alpha>0$ and $\beta>0$ are shape parameters. If $\beta<1$, the hazard function of this distribution has a bathtub shape, and has an increasing failure rate function, if $\beta\geq 1$. Note that, if $\alpha=1$, Equation (1) becomes an exponential power distribution. One may refer to Wu (2008), Rastogi, Tripathi, and Wu (2012), Ahmed (2014), Kayal et al. (2017) and the references cited there for some recent work done on the Chen distribution.

Stress-strength models have also been considered when a system consists of several components. Bhattacharyya and Johnson (1974) developed the multicomponent stress-strength model consisting of j components which functions when s $(1\leq s\leq j)$ or more of the components work simultaneously with a common random stress. This model corresponds to the s-out-of-j:G system and the reliability of this system can be written as $P(X_{j-s+1:j}>Z)$, where $X_1,\ldots,X_j$ are the strengths of the components from the CDF $F(x;\alpha,\beta)$ and Z is the common random stress from the CDF $F(z;\eta,\beta)$. Furthermore, $X_{j-s+1:j}$ is the $(j-s+1)$-th order statistic of $(X_1,\ldots,X_j)$. In this manner, the reliability of a multicomponent stress-strength model is given by

$$\begin{aligned}R_{s,j}&=P\left(\text{at least } s \text{of the } (X_1,\ldots,X_j)\text{exceed } Z\right)\\&=\sum_{i=s}^{j}\binom{j}{i}\int_{-\infty}^{\infty}(1-F(z;\alpha,\beta))^{i}(F(z;\alpha,\beta))^{j-i}dF(z;\eta,\beta)\end{aligned}$$

This system reliability was considered by Jae and Eun (1981) when both stress and strength follow a Weibull distribution with unknown scale parameters and known shape parameter. For some more literature on multicomponent stress-strength models, see Hanagal (1999, 2003) and Eryilmaz (2008, 2010).

Assume that $X_1,X_2,\ldots,X_j$ are independent random variables from Chen distribution with parameters (α,β), and Z is a random variable also from Chen distribution with parameters (η,β). For our model, $R_{s,j}$ can be written as

$$\begin{aligned}R_{s,j}&=\eta\beta\sum_{i=s}^{j}\binom{j}{i}\int_0^{\infty}z^{\beta-1}\exp\left[\left(1-e^{z^{\beta}}\right)(\alpha i+\eta)+z^{\beta}\right]\left[1-\exp\left(\alpha\left(1-e^{z^{\beta}}\right)\right]^{j-i}dz\\&=\eta\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^k\int_1^{\infty}\exp\left[\left(\alpha(i+k)+\eta\right)(1-t)\right]dt\\&=\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^k\frac{\eta}{\alpha(i+k)+\eta}\end{aligned}\tag{2}$$

where $t=e^{z^{\beta}}$.

3. Likelihood inference and information matrix

Suppose that n items are put into a life testing experiment and the observed data are $X_{i1}, X_{i2}, \ldots, X_{ij}$ and Z_i, $i = 1, 2, \ldots, n$. The likelihood equation can then be written as

$$L(\alpha, \eta, \beta; x, z) = \alpha^{nj}\eta^n\beta^{n(j+1)}\prod_{i=1}^{n}\prod_{k=1}^{j}\left\{x_{ik}^{\beta-1}\exp\left[\alpha\left(1-e^{x_{ik}^{\beta}}\right)+x_{ik}^{\beta}\right]\right\}$$
$$\prod_{i=1}^{n}\left\{z_i^{\beta-1}\exp\left[\eta\left(1-e^{z_i^{\beta}}\right)+z_i^{\beta}\right]\right\}$$

The log-likelihood function is given by

$$l(\alpha, \eta, \beta; x, z) = nj\ln\alpha + n\ln\eta + n(j+1)\ln\beta + (\beta-1)\sum_{i=1}^{n}\left(\sum_{k=1}^{j}\ln x_{ik} + \ln z_i\right)$$
$$+\sum_{i=1}^{n}\left(\sum_{k=1}^{j}x_{ik}^{\beta} + z_i^{\beta}\right) - \alpha V_{\beta} - \eta W_{\beta}$$

where $V_{\beta} = -\sum_{i=1}^{n}\sum_{k=1}^{j}(1-e^{x_{ik}^{\beta}})$ and $W_{\beta} = -\sum_{i=1}^{n}(1-e^{z_i^{\beta}})$. The MLEs of α and η can then be obtained as

$$\hat{\alpha} = \frac{nj}{V_{\hat{\beta}}} \quad \text{and} \quad \hat{\eta} = \frac{n}{W_{\hat{\beta}}},$$

where $\hat{\beta}$ is the MLE of β which can be obtained by solving the following non linear equation

$$\frac{n(j+1)}{\hat{\beta}} + \sum_{i=1}^{n}\left(\sum_{k=1}^{j}\ln x_{ik} + \ln z_i\right) + \sum_{i=1}^{n}\left(\sum_{k=1}^{j}x_{ik}^{\hat{\beta}}\ln x_{ik} + z_i^{\hat{\beta}}\ln z_i\right)$$
$$-\sum_{i=1}^{n}\left(\hat{\alpha}\sum_{k=1}^{j}e^{x_{ik}^{\hat{\beta}}}x_{ik}^{\hat{\beta}}\ln x_{ik} + \hat{\eta}e^{z_i^{\hat{\beta}}}z_i^{\hat{\beta}}\ln z_i\right) = 0$$

We can solve the above equation by using the Newton-Raphson method and then obtain $\hat{\beta}$.

The Fisher information matrix of $\theta = (\alpha, \eta, \beta)$ can be written as

$$\mathrm{I}(\theta) = -\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The elements of the matrix are obtained as

$$a_{11} = E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) = -\frac{nj}{\alpha^2},$$
$$a_{12} = a_{21} = E\left(\frac{\partial^2 l}{\partial\alpha\partial\eta}\right) = 0,$$
$$a_{13} = a_{31} = E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) = -\frac{nj\alpha}{\beta}\rho_1(\alpha),$$
$$a_{22} = E\left(\frac{\partial^2 l}{\partial\eta^2}\right) = -\frac{n}{\eta^2},$$
$$a_{23} = a_{32} = E\left(\frac{\partial^2 l}{\partial\eta\partial\beta}\right) = -\frac{n\eta}{\beta}\rho_1(\eta),$$
$$a_{33} = E\left(\frac{\partial^2 l}{\partial\beta^2}\right) = -\frac{n(j+1)}{\beta^2} + nj(\rho_2(\alpha)-\rho_3(\alpha)) + n(\rho_2(\eta)-\rho_3(\eta))$$

where $\rho_1(c)=\int_0^\infty(1+u)\ln(1+u)\ln(\ln(1+u))e^{-cu}\,du$, $\rho_2(c)=\frac{c}{\beta^2}\int_0^\infty\ln(1+u)[\ln(\ln(1+u))]^2e^{-cu}\,du$, and $\rho_3(c)=\frac{c^2}{\beta^2}\int_0^\infty(1+u)\ln(1+u)[\ln(\ln(1+u))]^2\,[1+\ln(1+u)]e^{-cu}\,du$.

4. Estimation of $R_{s,j}$

In this section, we will study the estimation of $R_{s,j}$ when β may be unknown or known.

4.1. β is unknown

In this section we derive the MLE and the Bayes estimator of $R_{s,j}$ under the assumption that the parameter β is unknown.

4.1.1. MLE of $R_{s,j}$

By invariance property of the MLE, we can obtain the MLE, $\hat{R}_{s,j}$, of $R_{s,j}$ by substituting the MLEs of (α,η,β) obtained in Section 3 into Equation (2). We next observe that $\hat{R}_{s,j}$ is asymptotically normal with mean $R_{s,j}$ and asymptotic variance

$$Var\left(\hat{R}_{s,j}\right)=\left(\frac{\partial R_{s,j}}{\partial\alpha}\right)^2I_{11}^{-1}+\left(\frac{\partial R_{s,j}}{\partial\eta}\right)^2I_{22}^{-1}+2\left(\frac{\partial R_{s,j}}{\partial\alpha}\right)\left(\frac{\partial R_{s,j}}{\partial\eta}\right)I_{12}^{-1} \tag{3}$$

where I_{ij}^{-1} is the $(i,\,j)$-th entry of the inverse of the Fisher information matrix, and

$$\frac{\partial R_{s,j}}{\partial\alpha}=\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^{k+1}\frac{\eta(i+k)}{[\eta+\alpha(i+k)]^2}$$

and

$$\frac{\partial R_{s,j}}{\partial\eta}=\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^{k}\frac{\alpha(i+k)}{[\eta+\alpha(i+k)]^2}$$

Therefore, the $100(1-p)\%$ confidence interval of $R_{s,j}$ is given by

$$\left(\hat{R}_{s,j}-q_{p/2}\sqrt{\hat{Var}\left(\hat{R}_{s,j}\right)},\hat{R}_{s,j}+q_{p/2}\sqrt{\hat{Var}\left(\hat{R}_{s,j}\right)}\right)$$

where $q_{p/2}$ is the upper $p/2$ quantile of the standard normal distribution and $\hat{Var}(\hat{R}_{s,j})$ is the value of $Var(\hat{R}_{s,j})$ computed at MLEs of parameters.

4.1.2. Bayesian estimation of $R_{s,j}$

In this section, we consider estimation of α, η and β using the Bayes approach and then derive the Bayes estimator of $R_{s,j}$. We assume that α, η, and β are independent random variables and their prior distributions are gamma with hyperparameters $(c_i,\,d_i)$, $i=1,\,2,\,3$, respectively. The density function of a gamma distribution, denoted as $Gamma(c_i,d_i)$, is

$$g(x) = \frac{d_i^{c_i}}{\Gamma(c_i)} x^{c_i - 1} e^{-xd_i}, \quad x>0$$

where $c_i>0$ and $d_i>0$, $i=1, 2, 3$. The joint posterior density function of α, η, and β can be written as

$$\pi(\alpha, \eta, \beta | x, z) = D^{-1} \alpha^{nj+c_1-1} \eta^{n+c_2-1} \beta^{n(j+1)+c_3-1} \left\{ \prod_{i=1}^{n} \prod_{k=1}^{j} x_{ik}^{\beta-1} \right\} \left\{ \prod_{i=1}^{n} z_i^{\beta-1} \right\}$$
$$\exp \left[\sum_{i=1}^{n} \sum_{k=1}^{j} x_{ik}^{\beta} + \sum_{i=1}^{n} z_i^{\beta} - \alpha V_\beta - \eta W_\beta - (\alpha d_1 + \eta d_2 + \beta d_3) \right]$$

where D is the normalizing constant. The Bayes estimator of $R_{s,j}$ under squared error loss is obtained as

$$\tilde{R}_{s,j}^{B} = \int_0^\infty \int_0^\infty \int_0^\infty R_{s,j} \pi(\alpha, \eta, \beta | x, z) \; d\alpha d\eta d\beta \tag{4}$$

Since the triple integral given in Equation (4) cannot be solve analytically, we can apply some suitable numerical methods to approximate it. We will apply the Lindley approximation method which was proposed by Lindley (1980) and the Metropolis-Hastings (MH) algorithm proposed by Metropolis et al. (1953) and Hastings (1970) to compute the Bayes estimate $\tilde{R}_{s,j}^{B}$.

4.1.3. Lindley approximation method

Here we compute the Bayes estimate of unknown parametric function $R_{s,j}$ by Lindley approximation method. Based on Lindley approximation, the approximate Bayes estimate of a parametric function $h(\alpha, \eta, \beta)$ under squared error loss is given by

$$\tilde{h}^{L} = h + (h_1 p_1 + h_2 p_2 + h_3 p_3 + p_4 + p_5)$$
$$+0.5[(\sigma_{11} l_{111} + 2\sigma_{12} l_{121} + 2\sigma_{13} l_{131} + 2\sigma_{23} l_{231} + \sigma_{22} l_{221} + \sigma_{33} l_{331})(h_1 \sigma_{11} + h_2 \sigma_{12} + h_3 \sigma_{13})$$
$$+(\sigma_{11} l_{112} + 2\sigma_{12} l_{122} + 2\sigma_{13} l_{132} + 2\sigma_{23} l_{232} + \sigma_{22} l_{222} + \sigma_{33} l_{332})(h_1 \sigma_{21} + h_2 \sigma_{22} + h_3 \sigma_{23})$$
$$+(\sigma_{11} l_{113} + 2\sigma_{12} l_{123} + 2\sigma_{13} l_{133} + 2\sigma_{23} l_{233} + \sigma_{22} l_{223} + \sigma_{33} l_{333})(h_1 \sigma_{31} + h_2 \sigma_{32} + h_3 \sigma_{33})] \tag{5}$$

where $p_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}$, $i=1, 2, 3$, $p_4 = h_{12}\sigma_{12} + h_{13}\sigma_{13} + h_{23}\sigma_{23}$, $p_5 = 0.5(h_{11}\sigma_{11} + h_{22}\sigma_{22} + h_{33}\sigma_{33})$, and σ_{ij} is the (i, j)-th entry of the inverse of the observed information matrix. Furthermore, $\rho_1 = \frac{c_1-1}{\alpha} - d_1, \rho_2 = \frac{c_2-1}{\eta} - d_2, \rho_3 = \frac{c_3-1}{\beta} - d_3$, and σ_{ik} is the element of the matrix $[-l_{ik}]^{-1}, i, k = 1, 2, 3$. Now, substituting $h = R_{s,j}$, all other quantities in Equation (5) have the following representations:

$$\tilde{h}^L = \tilde{R}^L_{s,j}, \quad h = R_{s,j}, \quad h_1 = \frac{\partial R_{s,j}}{\partial \alpha} = \sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k+1} \frac{\eta(i+k)}{[\eta + \alpha(i+k)]^2},$$

$$h_2 = \frac{\partial R_{s,j}}{\partial \eta} = \sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k} \frac{\alpha(i+k)}{[\eta + \alpha(i+k)]^2}, \quad h_3 = \frac{\partial R_{s,j}}{\partial \beta} = 0,$$

$$h_{11} = \frac{\partial^2 R_{s,j}}{\partial \alpha^2} = 2\sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k} \frac{\eta(i+k)^2}{[\eta + \alpha(i+k)]^3},$$

$$h_{12} = h_{21} = \frac{\partial^2 R_{s,j}}{\partial \alpha \partial \eta} = \sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k+1} \frac{(\alpha(i+k)-\eta)(i+k)}{[\eta + \alpha(i+k)]^3},$$

$$h_{22} = \frac{\partial^2 R_{s,j}}{\partial \eta^2} = 2\sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k+1} \frac{\alpha(i+k)}{[\eta + \alpha(i+k)]^3},$$

$$h_{13} = h_{31} = \frac{\partial^2 R_{s,j}}{\partial \alpha \partial \beta} = 0, \quad h_{23} = h_{32} = \frac{\partial^2 R_{s,j}}{\partial \eta \partial \beta} = 0, \quad h_{33} = \frac{\partial^2 R_{s,j}}{\partial \beta^2} = 0,$$

$$l_{11} = -\frac{nj}{\alpha^2}, \quad l_{13} = l_{31} = -\sum_{i=1}^{n}\sum_{k=1}^{j} e^{x_{ik}^{\beta}} x_{ik}^{\beta} \ln x_{ik}, \quad l_{22} = -\frac{n}{\eta^2},$$

$$l_{23} = l_{32} = -\sum_{i=1}^{n} e^{z_i^{\beta}} z_i^{\beta} \ln z_i, \quad l_{12} = l_{21} = 0,$$

$$l_{33} = -\frac{n(j+1)}{\beta^2} + \sum_{i=1}^{n} \left(\sum_{k=1}^{j} x_{ik}^{\beta} (\ln x_{ik})^2 + z_i^{\beta} (\ln z_i)^2 \right)$$
$$- \sum_{i=1}^{n} \left(\sum_{k=1}^{j} \alpha e^{x_{ik}^{\beta}} x_{ik}^{\beta} (\ln x_{ik})^2 \left(1 + x_{ik}^{\beta}\right) + \eta e^{z_i^{\beta}} z_i^{\beta} (\ln z_i)^2 \left(1 + z_i^{\beta}\right) \right),$$

$$l_{111} = \frac{2nj}{\alpha^3}, \quad l_{121} = l_{122} = l_{123} = l_{211} = l_{212} = l_{213} = l_{221} = l_{223} = l_{112} = l_{113} = 0,$$

$$l_{222} = \frac{2n}{\eta^3}, l_{133} = l_{331} = -\sum_{i=1}^{n}\sum_{k=1}^{j} e^{x_{ik}^{\beta}} x_{ik}^{\beta} (\ln x_{ik})^2 \left(1 + x_{ik}^{\beta}\right),$$

$$l_{233} = l_{332} = -\sum_{i=1}^{n} e^{z_i^{\beta}} z_i^{\beta} (\ln z_i)^2 \left(1 + z_i^{\beta}\right),$$

$$l_{333} = \frac{2n(j+1)}{\beta^3} + \sum_{i=1}^{n} \left(\sum_{k=1}^{j} x_{ik}^{\beta} (\ln x_{ik})^3 + z_i^{\beta} (\ln z_i)^3 \right) + \mathcal{V}_\beta + \mathcal{W}_\beta$$

where

$$\mathcal{V}_\beta = -\sum_{i=1}^{n} \sum_{k=1}^{j} e^{x_{ik}^{\beta}} x_{ik}^{\beta} \left(\ln x_{ik}^{\beta} \right)^3 \left\{ x_{ik}^{\beta} + x_{ik}^{\beta} \left(1 + x_{ik}^{\beta}\right) + \left(1 + x_{ik}^{\beta}\right) \right\}$$

and

$$\mathcal{W}_\beta = -\sum_{i=1}^{n} e^{z_i^{\beta}} z_i^{\beta} \left(\ln z_i^{\beta} \right)^3 \left\{ z_i^{\beta} + z_i^{\beta} \left(1 + z_i^{\beta}\right) + \left(1 + z_i^{\beta}\right) \right\}$$

All the quantities of unknown (α, η, β) are evaluated at the MLEs $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$.

4.1.4. MH algorithm

The MH algorithm is a general algorithm for Bayesian computations. Here we compute Bayes estimate of the reliability under squared error loss using this algorithm. We observe that the marginal posterior distributions of α and η are gamma distributions which can be used to generate respective posterior samples. However, the marginal posterior distribution of β cannot be obtained in a known form, and thus, it is relatively difficult to generate samples from this distribution. Here we approximate this marginal posterior density by a normal proposal distribution. The following algorithm is used to generate samples to obtain the Bayes estimate $\tilde{R}_{s,j}^{MH}$ of $R_{s,j}$.

Step 1. Choose an initial guess of (α, η, β), say $(\alpha_0, \eta_0, \beta_0)$.
Step 2. Generate β' from $N(\beta_{i-1}, \sigma^2)$ at the i-th iterative stage. The variance term σ^2 can be obtained from the variance-covariance matrix which is the inverse of the Fisher information matrix.
Step 3. Generate α' from $Gamma(nj + c_1, d_1 + v_{\beta'})$ and η' from $Gamma(n + c_2, d_2 + w_{\beta'})$, respectively.
Step 4. Compute $r = \min\{1, \frac{\pi(\alpha', \eta', \beta' | x, z)}{\pi(\alpha_{i-1}, \eta_{i-1}, \beta_{i-1} | x, z)}\}$.
Step 5. Generate a random number u from $U(0, 1)$.
Step 6. If $u \le r$, then $\alpha_i \leftarrow \alpha'$, $\eta_i \leftarrow \eta'$, and $\beta_i \leftarrow \beta'$.
Step 7. Compute $R_{s,j}^i$ at $(\alpha_i, \eta_i, \beta_i)$.
Step 8. Repeat Steps 2-7 B times and obtain the posterior sample $R_{s,j}^i, i = 1, 2, \ldots, B$.

We use this MH sample to compute the Bayes estimate and to construct the HPD interval of $R_{s,j}$. The $R_{s,j}$ can be estimated as

$$\tilde{R}_{s,j}^{MH} = \frac{1}{B - B_0} \sum_{i=B_0+1}^{B} R_{s,j}^i$$

where B_0 is the number in burn-in period. We can also employ the method proposed by Chen and Shao (1999) to compute the HPD interval of $R_{s,j}$.

4.2. β is known

In this section, we obtain the MLE, UMVUE, and Bayes estimator of the reliability $R_{s,j}$ under the assumption that parameter β is known, say β_1. Note that $R_{s,j}$ does not depend on β, and hence, $R_{s,j}$ remains the same as in β unknown case.

4.2.1. MLE of $R_{s,j}$

When β is known and say $\beta = \beta_1$, the likelihood function of (α, η) is given by

$$L(\alpha, \eta; x, z, \beta_1) = H(x, z, \beta_1)\alpha^{nj}\eta^n e^{-(\alpha\tilde{\varrho} + \eta\tilde{\varsigma})}$$

where $H(x, z, \beta_1) = \beta_1^{n(j+1)} \exp\left[\sum_{i=1}^n (\sum_{k=1}^j x_{ik}^{\beta_1} + z_i^{\beta_1})\right] \prod_{i=1}^n \prod_{k=1}^j x_{ik}^{\beta_1 - 1} \prod_{i=1}^n z_i^{\beta_1 - 1}$, $\tilde{\varrho} = -\sum_{i=1}^n \sum_{k=1}^j (1 - e^{x_{ik}^{\beta_1}})$ and $\tilde{\varsigma} = -\sum_{i=1}^n (1 - e^{z_i^{\beta_1}})$. The corresponding log-likelihood function can be written as

$$l(\alpha, \eta; x, z, \beta_1) = \ln H(x, z, \beta_1) + nj \ln \alpha + n \ln \eta - \alpha\tilde{\varrho} - \eta\tilde{\varsigma}$$

Subsequently, the MLEs of α and η can be obtained as

$$\hat{\alpha} = \frac{nj}{\tilde{\varrho}} \qquad \text{and} \qquad \hat{\eta} = \frac{n}{\tilde{\varsigma}}.$$

Using invariance property of MLE, one can obtain the MLE $\hat{R}_{s,j}$ of $R_{s,j}$ by substituting $\hat{\alpha}$ and $\hat{\eta}$ into Equation (2). Moreover, the asymptotic distribution of $\hat{R}_{s,j}$ is normal with mean $R_{s,j}$ and variance

$$Var\left(\hat{R}_{s,j}\right) = \left(\frac{\partial R_{s,j}}{\partial \alpha}\right)^2 \frac{\alpha^2}{nj} + \left(\frac{\partial R_{s,j}}{\partial \eta}\right)^2 \frac{\eta^2}{n}$$

where

$$\frac{\partial R_{s,j}}{\partial \alpha} = \sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k+1} \frac{\eta(i+k)}{\left[\eta + \alpha(i+k)\right]^2}$$

and

$$\frac{\partial R_{s,j}}{\partial \eta} = \sum_{i=s}^{j} \binom{j}{i} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{k} \frac{\alpha(i+k)}{\left[\eta + \alpha(i+k)\right]^2}$$

The $100(1-p)\%$ confidence interval of $R_{s,j}$ is now given by

$$\left(\hat{R}_{s,j} - q_{p/2}\sqrt{\hat{Var}\left(\hat{R}_{s,j}\right)}, \hat{R}_{s,j} + q_{p/2}\sqrt{\hat{Var}\left(\hat{R}_{s,j}\right)}\right)$$

where $q_{p/2}$ is the upper $p/2$ quantile of the standard normal distribution and $\hat{Var}(\hat{R}_{s,j})$ is value of $Var(\hat{R}_{s,j})$ computed at associated MLEs.

4.2.2. UMVUE of $R_{s,j}$

In this section, we obtain the UMVUE of stress-strength reliability $R_{s,j}$. Using the linearity property of the UMVUE, it suffices to find the UMVUE of parametric function $\nu(\alpha, \eta) = \frac{\eta}{\alpha(i+k)+\eta}$. Note that $(\tilde{\varrho}, \tilde{\varsigma})$ is a complete sufficient statistic for (α, η). It is also easy to show that the densities of $\tilde{\varrho}$ and $\tilde{\varsigma}$ are gamma distributions with parameters (nj, α) and (n, η), respectively. To derive the UMVUE of $R_{s,j}$, we need the following lemma:

Lemma 1. *Define*

$$\Psi(\tilde{\varrho}^*, \tilde{\varsigma}^*) = \begin{cases} 1, & \tilde{\varrho}^* > (i+k)\tilde{\varsigma}^* \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{\varrho}^* = e^{X_{11}^{\beta}} - 1$ *and* $\tilde{\varsigma}^* = e^{Z_1^{\beta}} - 1$. *Then,* $\Psi(\tilde{\varrho}^*, \tilde{\varsigma}^*)$ *is an unbiased estimator of* $\nu(\alpha, \eta)$.

Proof. Notice that $\tilde{\varrho}^*$ and $\tilde{\varsigma}^*$ are independent and follow exponential distributions with parameters α and η, respectively. Then, we can obtain that

$$
\begin{aligned}
E(\Psi(\tilde{\varrho}^*,\tilde{\varsigma}^*)) &= P\big(\tilde{\varrho}^*>(i+k)\tilde{\varsigma}^*\big)\\
&= \alpha\eta\int_0^{\infty}\int_0^{\frac{\varrho^*}{i+k}} e^{-\alpha\varrho^*}e^{-\eta\varsigma^*}\ d\varsigma^* d\varrho^*\\
&= \alpha\int_0^{\infty} e^{-\alpha\varrho^*}\left[1-e^{-\frac{\eta\varrho^*}{i+k}}\right]\ d\varrho^*\\
&= \alpha\left[\frac{1}{\alpha}-\frac{1}{\alpha+\eta/(i+k)}\right]\\
&= \frac{\eta}{\alpha(i+k)+\eta}.
\end{aligned}
$$

This completes the proof of the lemma.

Now, the UMVUE of $\nu(\alpha,\eta)$, say $\hat{\nu}(\alpha,\eta)$, can be obtained by using the Lehmann–Scheffé Theorem and it is given by

$$
\begin{aligned}
\hat{\nu}(\alpha,\eta) &= E\big(\Psi(\tilde{\varrho}^*,\tilde{\varsigma}^*)|\tilde{\varrho}=\varrho,\tilde{\varsigma}=\varsigma\big)\\
&= P\big(\tilde{\varrho}^*>(i+k)\tilde{\varsigma}^*|\tilde{\varrho}=\varrho,\tilde{\varsigma}=\varsigma\big)\\
&= \int_{\Phi}\int f^*_{\tilde{\varrho}^*|\tilde{\varrho}=\varrho}(\varrho^*|\varrho)f^*_{\tilde{\varsigma}^*|\tilde{\varsigma}=\varsigma}(\varsigma^*|\varsigma)\ d\varrho^* d\varsigma^*,
\end{aligned}
\tag{6}
$$

where $\Phi=\{(\varrho^*,\varsigma^*);0<\varrho^*<\varrho,0<\varsigma^*<\varsigma,\varrho^*>(i+k)\varsigma^*\}$. The double integral in Equation (6) can be discussed in three cases, That is, *Case (i)* $\varsigma(i+k)<\varrho$, *Case (ii)* $\varsigma(i+k)>\varrho$, and *Case (iii)* $\varsigma(i+k)=\varrho$. Using the result of Basirat, Baratpour, and Ahmadi (2015) (see also, Kizilaslan (2017)), we have

Case (i):

$$
\begin{aligned}
\hat{\nu}(\alpha,\eta) &= \frac{(n-1)(nj-1)}{\varrho\varsigma}\int_0^{\varsigma}\int_{\varsigma^*(i+k)}^{\varrho}\left(1-\frac{\varrho^*}{\varrho}\right)^{nj-2}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{n-2}\ d\varrho^* d\varsigma^*\\
&= \frac{n-1}{\varsigma}\int_0^{\varsigma}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{n-2}\left[1-\frac{\varsigma^*(i+k)}{\varrho}\right]^{nj-1} d\varsigma^*\\
&= \sum_{r=0}^{nj-1}(-1)^r\left[\frac{(i+k)\varsigma}{\varrho}\right]^r\frac{\binom{nj-1}{r}}{\binom{n+r-1}{r}}.
\end{aligned}
$$

Case (ii):

$$
\begin{aligned}
\hat{\nu}(\alpha,\eta) &= \frac{(n-1)(nj-1)}{\varrho\varsigma}\int_0^{\varrho}\int_0^{\frac{\varrho^*}{i+k}}\left(1-\frac{\varrho^*}{\varrho}\right)^{nj-2}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{n-2}\ d\varsigma^* d\varrho^*\\
&= \frac{nj-1}{\varrho}\int_0^{\varrho}\left(1-\frac{\varrho^*}{\varrho}\right)^{nj-2}\left[1-\left(1-\frac{\varrho^*}{(i+k)\varsigma}\right)^{n-1}\right]d\varrho^*\\
&= 1-\sum_{r=0}^{n-1}(-1)^r\left[\frac{\varrho}{\varsigma(i+k)}\right]^r\frac{\binom{n-1}{r}}{\binom{nj+r-1}{r}}.
\end{aligned}
$$

Case (iii):

$$\begin{aligned}\hat{\nu}(\alpha,\eta) &= \frac{(n-1)(nj-1)}{\varrho\varsigma}\int_0^{\varsigma}\int_{\varsigma^*(i+k)}^{\varrho}\left(1-\frac{\varrho^*}{\varrho}\right)^{nj-2}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{n-2} d\varrho^* d\varsigma^* \\ &= \frac{n-1}{\varsigma}\int_0^{\varsigma}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{n-2}\left[1-\frac{\varsigma^*(i+k)}{\varrho}\right]^{nj-1} d\varsigma^* \\ &= \frac{n-1}{\varsigma}\int_0^{\varsigma}\left(1-\frac{\varsigma^*}{\varsigma}\right)^{nj+n-3} d\varsigma^* \\ &= \frac{n-1}{nj+n-2}.\end{aligned}$$

Hence, the UMVUE of $R_{s,j}$ is now given by

$$\hat{R}^U_{s,j} = \sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^k\ \hat{\nu}(\alpha,\eta)$$

4.2.3. Bayesian estimates of $R_{s,j}$

Assume that α and η are independent and have gamma prior distributions with parameters (c_1, d_1) and (c_2, d_2), respectively. The joint posterior distribution of α and η turns out to be

$$\pi(\alpha,\eta|x,z,\beta_1) = \frac{(d_1+\varrho)^{nj+c_1}(d_2+\varsigma)^{n+c_2}}{\Gamma(nj+c_1)\Gamma(n+c_2)}\alpha^{nj+c_1-1}\eta^{n+c_2-1}\exp\left[-\left\{\alpha(d_1+\varrho)+\eta(d_2+\varsigma)\right\}\right], \quad \alpha>0, \eta>0$$

The Bayes estimator of $R_{s,j}$ under the squared error loss function is then obtained as

$$\tilde{R}^B_{s,j} = \sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^k\int_0^{\infty}\int_0^{\infty}\frac{\eta}{\alpha(i+k)+\eta}\pi(\alpha,\eta|x,z,\beta_1)\ d\alpha\ d\eta \tag{7}$$

Now, we consider the transformation $b=\frac{\eta}{\alpha(i+k)+\eta}$ and $b^*=\alpha(i+k)+\eta$. Then, $0<b<1, 0<b^*<\infty, \alpha=\frac{b^*(1-b)}{i+k}$, and $\eta=bb^*$. The Jacobian of this transformation is $-\frac{b^*}{i+k}$. After some simple algebra, Equation (7) can be rewritten as

$$\tilde{R}^B_{s,j} = \sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}(-1)^k\frac{(1-u)^{n+c_2}}{B(nj+c_1,n+c_2)}\int_0^1 b^{n+c_2}(1-b)^{nj+c_1-1}(1-bu)^{-t}\ db \tag{8}$$

where $u=1-\frac{(d_2+\varsigma)(i+k)}{d_1+\varrho}$ and $t=nj+c_1+c_2+n$. Following Gradshteyn and Ryzhik (1994), we rewrite Equation (8) as

$$\tilde{R}^B_{s,j} = \begin{cases}\displaystyle\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}\frac{(-1)^k(1-u)^{n+c_2}(n+c_2)}{t}\ {}_2F_1(t,n+c_2+1,t+1,u), & |u|<1, \\ \displaystyle\sum_{i=s}^{j}\binom{j}{i}\sum_{k=0}^{j-i}\binom{j-i}{k}\frac{(-1)^k(n+c_2)}{(1-u)^{nj+c_1}t}\ {}_2F_1\left(t,nj+c_1,t+1,\frac{u}{u-1}\right), & u<-1\end{cases}$$

where ${}_2F_1(\vartheta,\iota,\kappa,u)=\frac{1}{B(\iota,\kappa-\iota)}\int_0^1 y^{\iota-1}(1-y)^{\kappa-\iota-1}(1-uy)^{-\vartheta}dy$.

Next, we compute the Bayes estimates of stress-strength reliability using Lindley approximation method and MH procedure. Although, in this particular case we are able to obtain the Bayes estimator in an explicit form, still it is worth computing such estimates using some approximation methods as well. It helps in comparing exact estimates with approximate ones to judge how efficient the approximate procedures with the exact one in terms of some optimal criteria such as estimated risk.

4.2.4. Lindley approximation

In this case, Bayes estimator of a parametric function $h(\alpha, \eta)$ turns out to be

$$\tilde{h}^L = h + 0.5\left[\sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}\sigma_{ik} + \pi^*_{30}\sigma_{11}(h_1\sigma_{11} + h_2\sigma_{12}) + \pi^*_{21}H_{12} + \pi^*_{21}H_{21} + \pi^*_{03}\sigma_{22}(h_2\sigma_{22} + h_1\sigma_{21})\right]$$

where $H_{ik} = 3h_i\sigma_{ii} + \sigma_{ik} + h_k\sigma_{ii}(\sigma_{ii} + 2\sigma^2_{ik})$. Furthermore, let

$$\pi^* \propto (nj + c_1 - 1)\ln\alpha + (n + c_2 - 1)\ln\eta - \alpha(d_1 + \varrho) - \eta(d_2 + \varsigma).$$

That is, π^* is proportional to the posterior density of (α, η). The posterior modes of α and η can be obtained from π^* and are given, respectively, by

$$\tilde{\alpha}_L = \frac{nj + c_1 - 1}{d_1 + \varrho} \quad \text{and} \quad \tilde{\eta}_L = \frac{n + c_2 - 1}{d_2 + \varsigma}.$$

Moreover, $\pi^*_{ij} = \frac{\partial^{i+j}\pi^*}{\partial\alpha^i\partial\eta^j}\big|_{\alpha=\tilde{\alpha}_L, \eta=\tilde{\eta}_L}$, $i, j = 0, 1, 2, 3$ and $i + j = 3$. Thus,

$$\pi^*_{30} = \frac{2(nj + c_1 - 1)}{\tilde{\alpha}_L^3},$$
$$\pi^*_{12} = 0 = \pi^*_{21},$$
$$\pi^*_{03} = \frac{2(n + c_2 - 1)}{\tilde{\eta}_L^3}$$

Finally, we take $h = R_{s,j}$ in above calculations and then obtain the Bayes estimate $\tilde{R}^L_{s,j}$ of $R_{s,j}$. Note that all computations are performed at $(\tilde{\alpha}_L, \tilde{\eta}_L)$.

4.2.5. MH algorithm

Here we use the MH algorithm to compute the Bayes estimate of $R_{s,j}$. Observe that the marginal posterior distributions of α and η follows gamma distributions. We can use the following algorithm to generate samples.

Step 1. Choose an initial guess of (α, η), say (α_0, η_0).
Step 2. Generate α' from $Gamma(n + c_1, d_1 + \varrho)$ and η' from $Gamma(n + c_2, d_2 + \varsigma)$.
Step 3. Compute $r = \min\{1, \frac{\pi(\alpha', \eta', \beta_1|x, z)}{\pi(\alpha_{i-1}, \eta_{i-1}, \beta_1|x, z)}\}$.
Step 4. Generate u from $U(0, 1)$.
Step 5. If $u \le r$, then $\alpha_i \leftarrow \alpha'$ and $\eta_i \leftarrow \eta'$.
Step 6. Compute $R^i_{s,j}$ at (α_i, η_i).

Step 7. Repeat Steps 2-6 B times and obtain the posterior sample $R^i_{s,j}, i = 1, 2, ..., B$.

This sample is then used to compute the Bayes estimate and to construct the HPD credible interval for $R_{s,j}$. Subsequently, $R_{s,j}$ can be estimated as

$$\tilde{R}^{MH}_{s,j} = \frac{1}{B - B_0} \sum_{i=B_0+1}^{B} R^i_{s,j},$$

where B_0 is the number in burn-in period. Further, we can construct the $100(1-p)\%$ HPD credible interval of $R_{s,j}$ using the method of Chen and Shao (1999).

5. Simulation study

In this section, we study the performance of different estimates of $R_{s,j}$ using Monte Carlo simulations. We generate samples from Chen distribution for different sample sizes. The performances of point estimators are compared using the estimated risks based on 4000 replications. Also, the performances of the confidence intervals and corresponding coverage probabilities are reported. The estimated risk in estimating θ with the estimator $\hat{\theta}$ is given by

$$ER(\theta) = \frac{1}{K} \sum_{i=1}^{K} \left(\hat{\theta}_i - \theta\right)^2.$$

First, we consider the case where all parameters (α, η, β) are unknown. We generate the stress and strength samples for $(\alpha, \eta, \beta) = (1.5, 2, 0.5), (2.5, 1.5, 0.75)$, and for different sample sizes $n = 10, 15, 20, ..., 50$. The corresponding true values of reliability in a multicomponent stress-strength with the given combinations $(s, j) = (1, 4)$ are 0.8665 and 0.6516, and for $(s, k) = (3, 6)$ these values are 0.6646 and 0.4118, respectively. We consider the following values of hyperparameters of the priors: $(c_1, d_1) = (3, 2), (c_2, d_2) = (8, 4), (c_3, d_3) = (4, 8)$ for the first set of true parameter values, and for the second set of true parameter values, we take the values of hyperparameters as $(c_1, d_1) = (5, 2), (c_2, d_2) = (6, 4), (c_3, d_3) = (3, 4)$. The simulation results of MLEs, Lindley estimates, MH estimates, and interval estimations of $R_{s,j}$ are reported in Tables 1 and 2. All the results were computed over 4000 simulated samples.

In Table 1, for each sample size n and estimator, the first value represents the average estimate of $R_{s,j}$ and the second value represents the estimated risk for the corresponding stress-strength reliability. These values are tabulated for various sample sizes. From Table 1, we observe that, in general, better estimation results may be obtained with an increase in the sample size. However, it is relatively difficult to observe in exact sense that estimated risks decreases with sample size using numerical simulations. We observe such behavior for both maximum likelihood and Bayes estimates. The Bayes estimates of $R_{s,j}$ under the squared error loss perform relatively better than the MLEs as far as the estimated risks are concerned. Among the Bayes estimates, we observe that the estimated risks of the Bayes estimates using Lindley approximation are generally smaller than those using the MH algorithm. From tabulated values, we observe that the Bayes estimates and their estimated risks marginally remain close to each other, as expected.

Table 1. Average estimates & estimated risks of estimators of $R_{s,j}$ when the parameter β is unknown.

n	$R_{1,4}$	$\hat{R}_{s,j}$	$\tilde{R}^{L}_{s,j}$	$\tilde{R}^{MH}_{s,j}$	$R_{3,6}$	$\hat{R}_{s,j}$	$\tilde{R}^{L}_{s,j}$	$\tilde{R}^{MH}_{s,j}$
10	0.8665	0.8549	0.8682	0.8611	0.6646	0.6946	0.6907	0.6532
		0.0052	0.0025	0.0025		0.0120	0.0036	0.0047
15		0.8757	0.8718	0.8616		0.6882	0.6869	0.6565
		0.0033	0.0017	0.0019		0.0080	0.0025	0.0039
20		0.8545	0.8601	0.8597		0.6741	0.6721	0.6590
		0.0026	0.0014	0.0016		0.0058	0.0024	0.0032
25		0.8556	0.8617	0.8607		0.6728	0.6710	0.6597
		0.0021	0.0013	0.0014		0.0046	0.0023	0.0028
30		0.8564	0.8629	0.8627		0.6719	0.6729	0.6609
		0.0018	0.0012	0.0013		0.0041	0.0023	0.0027
35		0.8580	0.8627	0.8624		0.6708	0.6717	0.6612
		0.0015	0.0010	0.0011		0.0034	0.0021	0.0024
40		0.8705	0.8635	0.8631		0.6692	0.6677	0.6618
		0.0014	0.0010	0.0010		0.0028	0.0019	0.0020
45		0.8587	0.8601	0.8640		0.6680	0.6671	0.6627
		0.0011	0.0009	0.0009		0.0025	0.0017	0.0019
50		0.8598	0.8606	0.8656		0.6676	0.6662	0.6634
		0.0010	0.0008	0.0008		0.0024	0.0017	0.0018
10	0.6516	0.6286	0.6398	0.6479	0.4118	0.4327	0.4314	0.3991
		0.0131	0.0052	0.0061		0.0117	0.0032	0.0045
15		0.6327	0.6376	0.6477		0.4028	0.4014	0.4049
		0.0084	0.0041	0.0048		0.0068	0.0027	0.0034
20		0.6329	0.6400	0.6489		0.4035	0.4021	0.4063
		0.0062	0.0036	0.0040		0.0054	0.0027	0.0032
25		0.6361	0.6432	0.6462		0.4049	0.4033	0.4088
		0.0051	0.0034	0.0036		0.0042	0.0025	0.0028
30		0.6374	0.6453	0.6475		0.4057	0.4037	0.4092
		0.0043	0.0030	0.0032		0.0034	0.0022	0.0024
35		0.6419	0.6451	0.6567		0.4084	0.4052	0.4108
		0.0037	0.0027	0.0028		0.0030	0.0021	0.0022
40		0.6454	0.6599	0.6559		0.4091	0.4077	0.4120
		0.0032	0.0024	0.0025		0.0027	0.0020	0.0021
45		0.6554	0.6520	0.6537		0.4108	0.4113	0.4117
		0.0029	0.0023	0.0023		0.0022	0.0017	0.0018
50		0.6507	0.6552	0.6543		0.4116	0.4112	0.4120
		0.0025	0.0020	0.0020		0.0021	0.0017	0.0017

In Table 2, we present the 95% asymptotic confidence intervals and HPD intervals along with their coverage probabilities. From Table 2, it can be observed that average lengths of HPD intervals are smaller than those of the asymptotic confidence intervals. The coverage probabilities of both intervals are relatively satisfactory (see also, Kizilaslan (2017)). We have also observed similar behavior of proposed point and interval estimates for some other parameter values such as $(\alpha, \eta, \beta) = (0.5, 1, 0.5)$ using Monte Carlo simulations. Results are not presented here for the sake of conciseness.

Next, we consider the case when β is known. We generate stress and strength samples for previously selected values of (α, η) and hyperparameters when $\beta = 0.25$. The simulation results are reported in Tables 3 and 4. All estimates are computed over 4000 simulated samples.

In Table 3, average estimates and estimated risks of MLEs, UMVUEs, exact Bayes estimates, Lindley estimates, and MH estimates of $R_{s,j}$ are presented when $\beta = 0.25$. From this table, we observe that the estimated risk decreases when sample size increase. We further observe that MLEs and UMVUEs compete well with Bayes estimates.

Table 2. Average lengths (ALs) and coverage probabilities (CPs) of the interval estimations for $R_{s,j}$ when β is unknown.

		Asymptotic		HPD			Asymptotic		HPD	
n	$R_{1,4}$	AL	CP	AL	CP	$R_{3,6}$	AL	CP	AL	CP
10	0.8665	0.2668	0.87	0.1563	0.89	0.6646	0.4032	0.90	0.2261	0.90
15		0.2233	0.90	0.1354	0.88		0.3357	0.91	0.1969	0.88
20		0.1963	0.91	0.1215	0.87		0.2943	0.93	0.1777	0.87
25		0.1767	0.92	0.1114	0.86		0.2646	0.93	0.1630	0.87
30		0.1613	0.92	0.1028	0.85		0.2424	0.92	0.1521	0.85
35		0.1494	0.92	0.0962	0.86		0.2252	0.94	0.1425	0.85
40		0.1403	0.93	0.0910	0.84		0.2114	0.94	0.1349	0.86
45		0.1326	0.93	0.0864	0.85		0.1994	0.94	0.1281	0.86
50		0.1258	0.93	0.0823	0.85		0.1894	0.94	0.1225	0.84
10	0.6516	0.4208	0.90	0.2527	0.89	0.4118	0.3881	0.92	0.2238	0.90
15		0.3489	0.92	0.2174	0.87		0.3183	0.94	0.1927	0.88
20		0.3051	0.93	0.1938	0.87		0.2767	0.94	0.1724	0.87
25		0.2740	0.93	0.1767	0.86		0.2471	0.94	0.1567	0.85
30		0.2508	0.93	0.1635	0.84		0.2263	0.94	0.1454	0.86
35		0.2332	0.94	0.1534	0.84		0.2099	0.94	0.1361	0.84
40		0.2182	0.94	0.1442	0.84		0.1966	0.94	0.1283	0.84
45		0.2060	0.94	0.1368	0.84		0.1850	0.95	0.1216	0.84
50		0.1956	0.94	0.1305	0.84		0.1759	0.94	0.1163	0.83

However, Bayes estimates have an advantage over these two estimates in terms of estimated risk. Results also indicate that MLEs perform marginally better than UMVUEs. Further, the exact Bayes estimates are very close to the Bayes estimates which are obtained by using Lindley approximation and MH methods. Besides, Lindley and MH estimates deviate marginally from exact estimates and for large sample sizes, their estimated risks tend to become close to each other.

Table 4 contains the average lengths of asymptotic confidence intervals and HPD intervals along with their coverage probabilities. It is observed that, in general, the average length of HPD intervals are shorter than asymptotic confidence intervals. We observe that the lengths of both intervals decrease when sample size increases. The coverage probabilities of these intervals are relatively satisfactory (see also, Kizilaslan (2017)). We also computed these estimates using some other arbitrary combinations of (α, η) such as (1.5,2), (2.5,1.5), and (0.5,1) when the parameter β may be 0.85, 0.15, or 0.5. We draw quite similar conclusions from these tables as well. However, these results are not presented here for the sake of brevity.

6. Data analysis

We consider a data set which was initially published in Musa (1979) and discussed in Nikora and Lyu (1996). This data set is also available at http://www.cse.cuhk.edu.hk/~lyu/book/reliability/DATA/CH7/SYS2.DAT and represents failure times of different subjects under a study. Here we consider $s=3$ and $j=6$ which suggests that it is a 3-out-of-6:G system. Let Z_1 denotes the 17-th failure time and $X_{1j}, j = 1, 2, ..., 6$, be the times to failure of observations numbered 18 to 23. Similarly, let Z_2 be the failure time of the 24-th observation and $X_{2j}, j = 1, 2, ..., 6$, be the failure times of observations lying between 25 to 30. When we carry on this data process up to 51-st failure, then we get $n=5$. The data (X, Z) are as follows:

Table 3. Average estimates and estimated risks of estimators of $R_{s,j}$ when $\beta = 0.25$.

n	$R_{1,4}$	$\hat{R}_{s,j}$	$\hat{R}^{U}_{s,j}$	$\tilde{R}^{B}_{s,j}$	$\tilde{R}^{L}_{s,j}$	$\tilde{R}^{MH}_{s,j}$	$R_{3,6}$	$\hat{R}_{s,j}$	$\hat{R}^{U}_{s,j}$	$\tilde{R}^{B}_{s,j}$	$\tilde{R}^{L}_{s,j}$	$\tilde{R}^{MH}_{s,j}$
10	0.8665	0.8779	0.8787	0.8576	0.8572	0.8575	0.6646	0.6955	0.6739	0.6774	0.6745	0.6777
		0.0019	0.0022	0.0007	0.0007	0.0007		0.0029	0.0031	0.0008	0.0008	0.0008
15		0.8706	0.8702	0.8596	0.8594	0.8618		0.6802	0.6750	0.6706	0.6716	0.6727
		0.0014	0.0016	0.0007	0.0007	0.0007		0.0019	0.0020	0.0008	0.0008	0.0008
20		0.8717	0.8733	0.8636	0.8634	0.8636		0.6603	0.6557	0.6591	0.6590	0.6601
		0.0012	0.0013	0.0006	0.0006	0.0007		0.0017	0.0016	0.0008	0.0008	0.0008
25		0.8685	0.8724	0.8633	0.8642	0.8640		0.6604	0.6597	0.6691	0.6696	0.6701
		0.0010	0.0010	0.0006	0.0006	0.0006		0.0013	0.0013	0.0008	0.0007	0.0008
30		0.8702	0.8708	0.8659	0.8658	0.8666		0.6703	0.6582	0.6578	0.6645	0.6573
		0.0010	0.0010	0.0006	0.0006	0.0006		0.0012	0.0012	0.0007	0.0007	0.0008
35		0.8682	0.8701	0.8650	0.8649	0.8655		0.6626	0.6609	0.6687	0.6696	0.6705
		0.0008	0.0009	0.0006	0.0006	0.0006		0.0012	0.0012	0.0006	0.0006	0.0006
40		0.8699	0.8710	0.8622	0.8612	0.8618		0.6590	0.6576	0.6597	0.6609	0.6599
		0.0008	0.0008	0.0006	0.0006	0.0006		0.0011	0.0011	0.0005	0.0006	0.0005
45		0.8626	0.8606	0.8653	0.8634	0.8649		0.6681	0.6669	0.6635	0.6639	0.6638
		0.0007	0.0007	0.0006	0.0006	0.0006		0.0011	0.0010	0.0005	0.0005	0.0005
50		0.8650	0.8639	0.8652	0.8652	0.8657		0.6680	0.6593	0.6629	0.6621	0.6631
		0.0007	0.0007	0.0006	0.0005	0.0005		0.0010	0.0010	0.0004	0.0004	0.0004
10	0.6726	0.6569	0.6581	0.6660	0.6641	0.6436	0.4118	0.3966	0.3945	0.4047	0.4041	0.4010
		0.0023	0.0024	0.0008	0.0008	0.0008		0.0022	0.0023	0.0008	0.0008	0.0008
15		0.6484	0.6821	0.6672	0.6648	0.6637		0.4284	0.4167	0.4198	0.4187	0.4064
		0.0017	0.0017	0.0008	0.0008	0.0008		0.0016	0.0016	0.0008	0.0008	0.0008
20		0.6615	0.6634	0.6682	0.6685	0.6669		0.4267	0.4173	0.4189	0.4179	0.4193
		0.0014	0.0014	0.0008	0.0008	0.0008		0.0013	0.0013	0.0008	0.0008	0.0008
25		0.6635	0.6642	0.6692	0.6682	0.6689		0.4156	0.4082	0.4182	0.4169	0.4188
		0.0012	0.0012	0.0008	0.0008	0.0008		0.0010	0.0010	0.0008	0.0006	0.0006
30		0.6669	0.6672	0.6693	0.6997	0.6691		0.4170	0.4118	0.4144	0.4136	0.4120
		0.0011	0.0011	0.0007	0.0007	0.0008		0.0011	0.0010	0.0006	0.0006	0.0006
35		0.6687	0.6677	0.6703	0.6712	0.6705		0.4274	0.4222	0.4337	0.4236	0.4212
		0.0010	0.0010	0.0007	0.0007	0.0007		0.0010	0.0010	0.0004	0.0005	0.0005
40		0.6689	0.6678	0.6702	0.6702	0.6718		0.4316	0.4270	0.4248	0.4215	0.4265
		0.0010	0.0010	0.0007	0.0006	0.0006		0.0010	0.0009	0.0004	0.0004	0.0004
45		0.6694	0.6686	0.6709	0.6722	0.6727		0.4270	0.4230	0.4201	0.4226	0.4208
		0.0009	0.0010	0.0006	0.0006	0.0006		0.0009	0.0009	0.0003	0.0004	0.0003
50		0.6697	0.6684	0.6726	0.6729	0.6629		0.4354	0.4377	0.4093	0.4105	0.4129
		0.0010	0.0010	0.0005	0.0004	0.0004		0.0009	0.0008	0.0003	0.0003	0.0003

Table 4. Average lengths (ALs) and coverage probabilities (CPs) of the interval estimations when $\beta = 0.25$.

		Asymptotic		HPD			Asymptotic		HPD	
n	$R_{1,4}$	AL	CP	AL	CP	$R_{3,6}$	AL	CP	AL	CP
10	0.8665	0.2547	0.86	0.1507	0.88	0.6646	0.4161	0.91	0.2289	0.89
15		0.2144	0.91	0.1312	0.91		0.3426	0.92	0.1999	0.91
20		0.1891	0.89	0.1182	0.86		0.2977	0.94	0.1795	0.89
25		0.1715	0.97	0.1087	0.90		0.2673	0.94	0.1648	0.91
30		0.1570	0.92	0.1008	0.91		0.2441	0.92	0.1531	0.89
35		0.1469	0.93	0.0950	0.89		0.2262	0.91	0.1436	0.89
40		0.1377	0.95	0.0900	0.89		0.2120	0.95	0.1357	0.91
45		0.1301	0.96	0.0852	0.89		0.2000	0.94	0.1291	0.88
50		0.1239	0.92	0.0814	0.84		0.1899	0.96	0.1233	0.88
10	0.6516	0.4280	0.93	0.2464	0.92	0.4118	0.3911	0.97	0.2183	0.91
15		0.3513	0.97	0.2123	0.94		0.3186	0.96	0.1886	0.91
20		0.3052	0.94	0.1901	0.90		0.2763	0.91	0.1691	0.88
25		0.2735	0.94	0.1733	0.90		0.2486	0.94	0.1554	0.84
30		0.2498	0.97	0.1603	0.89		0.2290	0.98	0.1451	0.89
35		0.2314	0.96	0.1498	0.89		0.2116	0.98	0.1355	0.97
40		0.2168	0.96	0.1416	0.89		0.1985	0.96	0.1281	0.94
45		0.2045	0.95	0.1342	0.89		0.1883	0.96	0.1223	0.95
50		0.1940	0.97	0.1277	0.89		0.1783	0.93	0.1166	0.91

Table 5. Goodness of fit for the data X and Z separately.

Distribution	Data X				Data Z			
	$\hat{\alpha}$	$\hat{\beta}$	K-S	p-value	$\hat{\eta}$	$\hat{\beta}$	K-S	p-value
Bathtub	0.0084	0.2245	0.1442	0.60	0.0023	0.2922	0.2130	0.90
Inverse Weibull	0.5707	25.8900	0.2516	0.04	1.7360	15157	0.2181	0.90
Exponential		0.0010	0.1536	0.15		0.0024	0.2852	0.70

$$X = \begin{bmatrix} 277 & 437 & 437 & 596 & 757 & 2230 \\ 277 & 363 & 405 & 522 & 535 & 613 \\ 213 & 298 & 821 & 1300 & 1601 & 1620 \\ 5 & 149 & 618 & 1034 & 2441 & 2640 \\ 437 & 565 & 714 & 927 & 1119 & 4462 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 135 \\ 340 \\ 277 \\ 874 \\ 460 \end{bmatrix}.$$

We first verify that the Chen distribution can be used to analyze the given data set. In Table 5, we obtain the MLEs of unknown parameters of all the three competing models including inverse Weibull and exponential distributions with respect to both the data sets. We also report the Kolmogorov-Smirnov (K-S) statistics along with corresponding p-values. From this table, we observe that the Chen distribution provides quite good fit to the given data set compared to the other distributions.

We obtain the MLEs of (α, η, β) by using the formulae in Section 3. They are $\hat{\alpha} = 0.0070, \hat{\eta} = 0.0186$, and $\hat{\beta} = 0.2289$. Next, we obtain the MLE and 95% asymptotic confidence interval for reliability $R_{s,j}$ as $\hat{R}_{s,j} = 0.8549$ and $(0.6385, 1)$, respectively. The upper limit of the asymptotic confidence interval is equal to 1 because its calculated value is 1.0710 and the reliability must be less than or equal to 1. We mention that the Bayes estimates are computed using the non-informative prior distribution. The Bayes estimates of $R_{s,j}$ by using Lindley approximation and MH algorithm are $\tilde{R}^{L}_{s,j} = 0.8408$ and $\tilde{R}^{MH}_{s,j} = 0.8355$, respectively. The 95% HPD interval for $R_{s,j}$ is $(0.6670, 0.9713)$.

7. Conclusions

In this article, we study the multicomponent stress-strength reliability when both stress and strength variables follow the Chen distribution. We use classical and Bayesian approaches to obtain the estimates of the reliability $R_{s,j}$ when common parameter β may be known or unknown. The exact Bayes estimate is also obtained when β is known. We compute the Bayes estimates of $R_{s,j}$ using Lindley approximation and HM algorithm. We also obtain the UMVUE of $R_{s,j}$ with known β and compare its performance with other proposed estimates. Based on the simulation results, we observe that, in general, estimated risks of proposed estimators of $R_{s,j}$ show good behavior with an increase in sample size. However, exact behavior that risk strictly decreases with sample size may not be observed using numerical simulations. In general, the average lengths of the intervals tend to decrease as sample size increases. From tabulated results, we find that estimated risks of Bayes estimates generally remain smaller than risks of the other proposed procedures.

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